A Phase-Space Approach to FOM Reverberation/Clutter Models

Leon Cohen

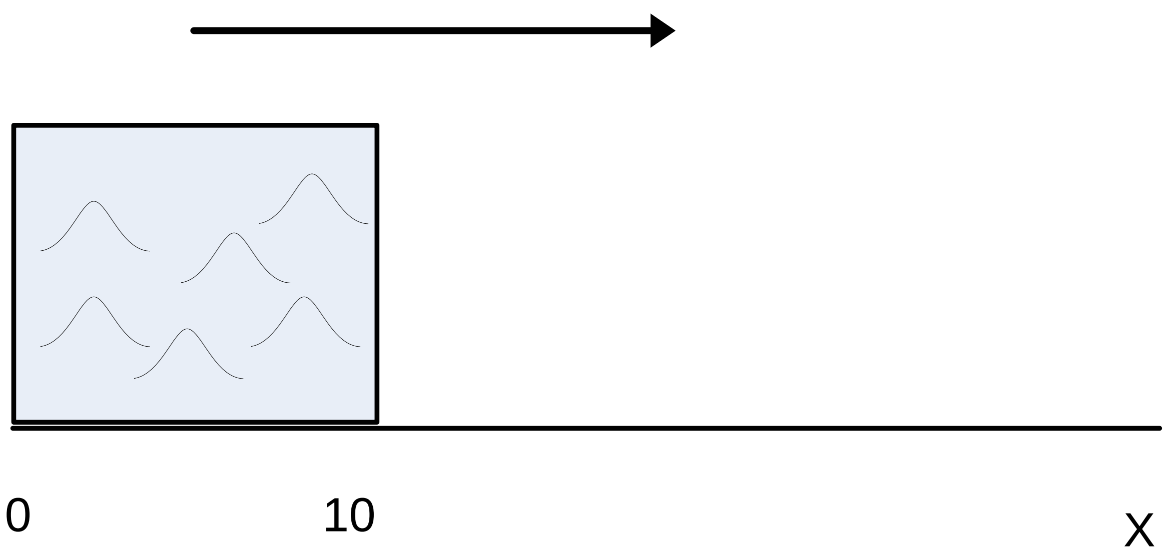
FOM noise models

- The noise is the sum of elementary signals,

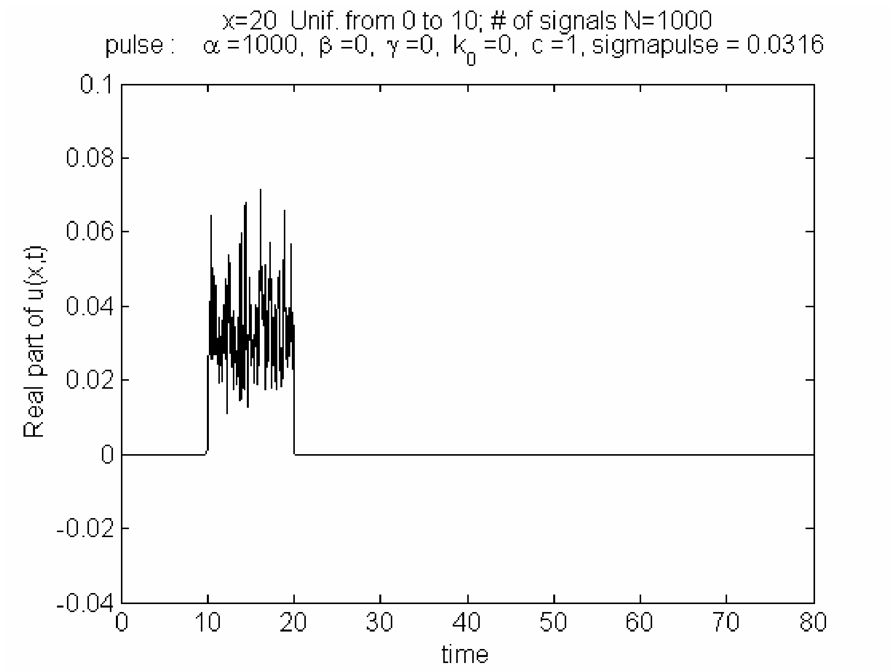
$$s(t) = \sum_{n=1}^{N} a_n u_n(t; \varepsilon_n)$$

- u_n : deterministic (the "elementary signal") related to the inducing pulse.
- a_n and ε_n : random parameters
- Sometimes called FOM because of the fundamental papers of Faure, Ol'shevskii, and Middleton
- However, such models for noise have a long history:

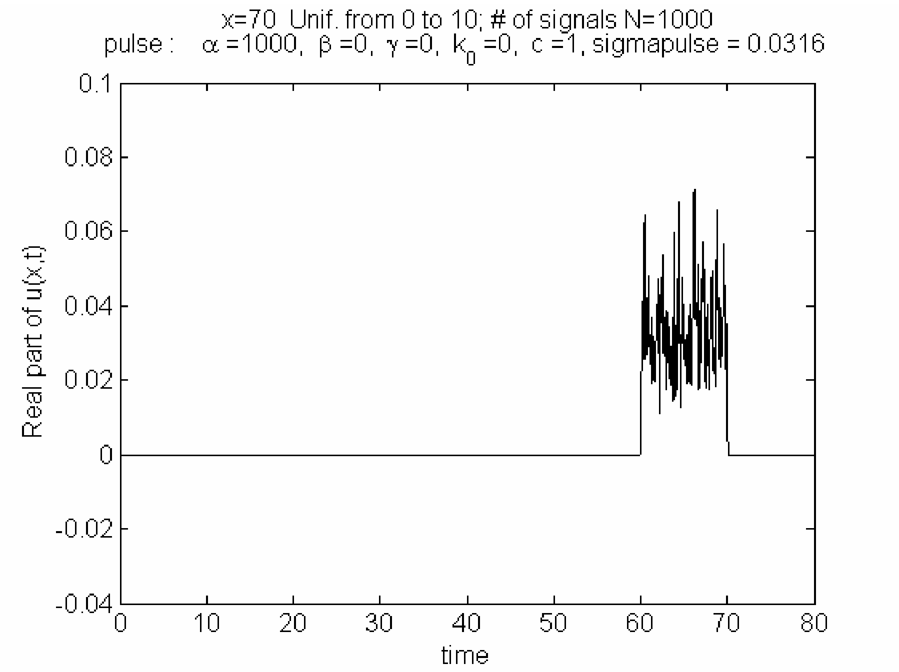
Shot noise – Generalized:
Pulses with width and dispersive propagation



“Moving” noise generated at (0,10) at time zero ; c=1
1000 random Gaussian pulses ; No dispersion



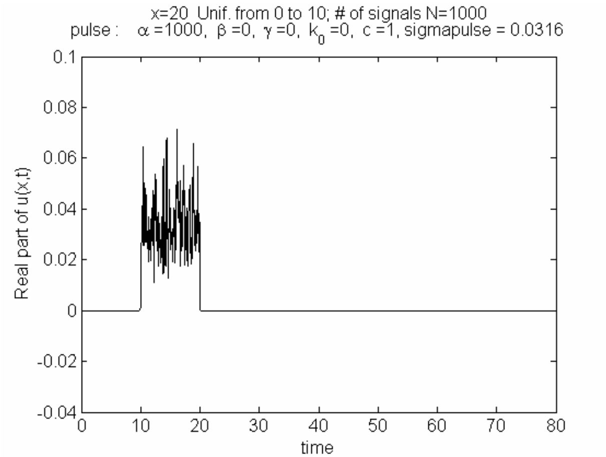
At x=20



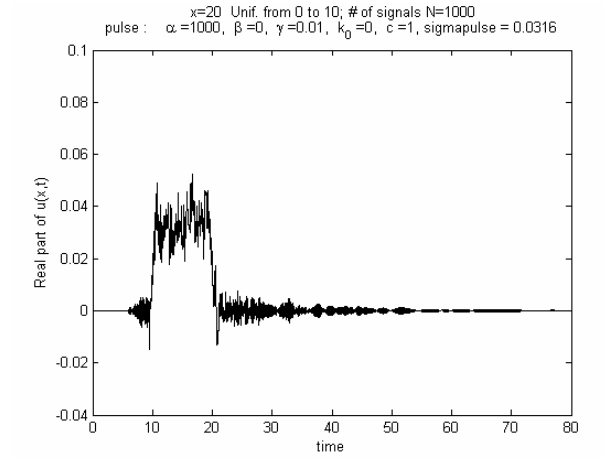
At x=70

Noise propagation with dispersion

No dispersion:



A bit of dispersion:

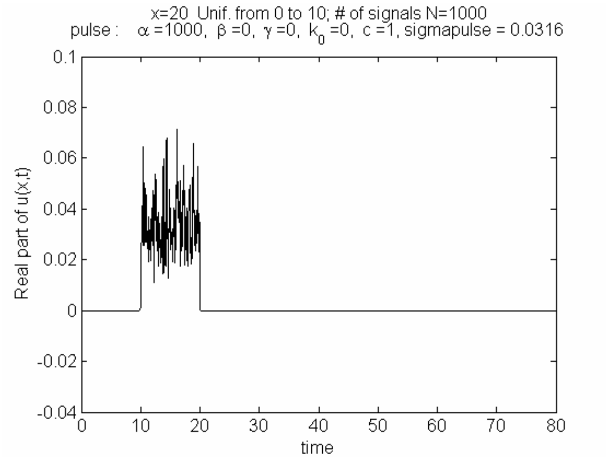


more:

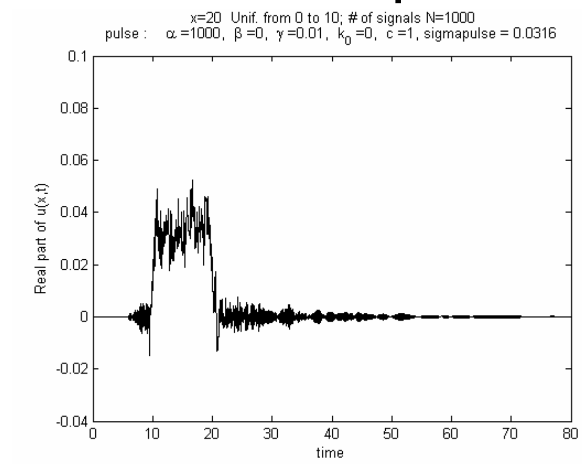
even more:

Noise propagation with dispersion

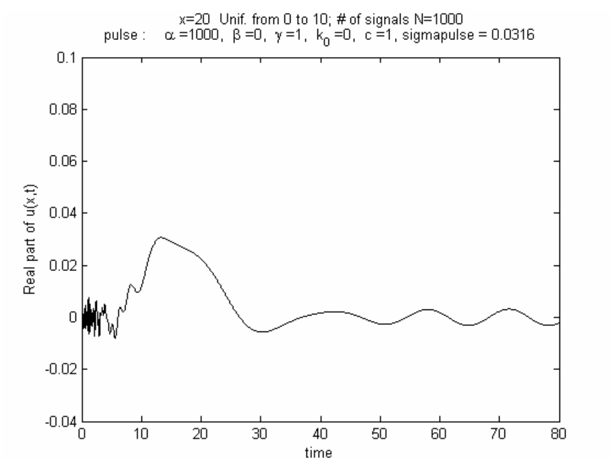
No dispersion:



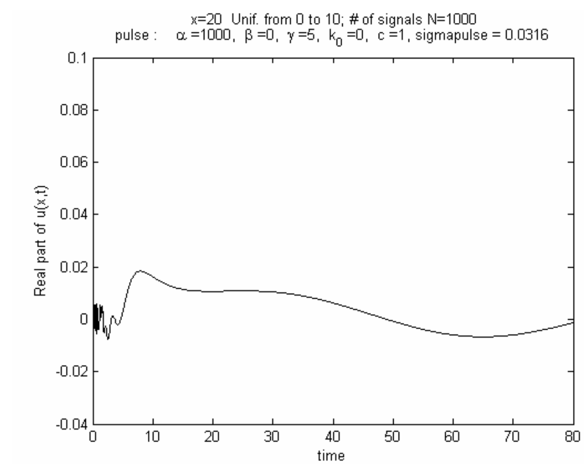
A bit of dispersion:



more:



even more:

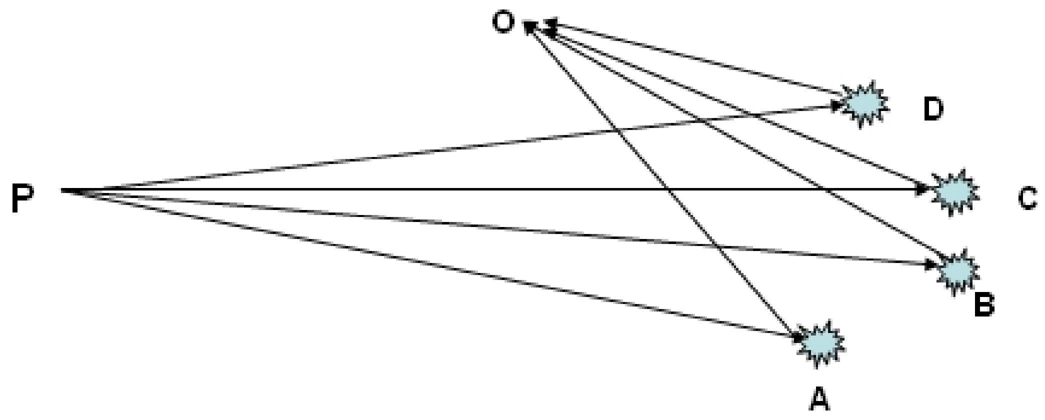


Shot noise – Generalized:
Pulses with width and dispersion propagation

P: projector

A,B,C,D,…: scatterers

O: observation point



“Heat bath” models of noise

Noise calculations of this type started with:

- Pearson and Raleigh – sort of – Pearson proposes random walk (1905) and Raleigh says he did it for another reason in 1880 “the composition of n isoperiodic vibrations of unit amplitude and phases distributed at random”
- Markoff (1912), Holtsmark (1919), Chandrasekhar (1943)
- Particularly important: Ford, Kac, and Mazur — 1965: derived from first principle “white noise” in the Langevin equation

Induced Noise

- The “noise” is induced by a pulse or train of pulses
- Historically: the problem was recognized to be highly nonstationary with many facets
- A basic argument is that things are so random – that they are:
- *Super* random: Rayleigh (= Gaussian) and modifications

Super Random: Gaussian

→ Rayleigh

$$s = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n e^{iS_n} \qquad \text{random phasor sum}$$

$$R = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \cos S_n \qquad \text{(real part)}$$

$$I = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \sin S_n \qquad \text{(imaginary part)}$$

Probability of R: Gaussian
Probability of I: Gaussian

$$P(R) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-R^2/2\sigma^2} \qquad P(I) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-I^2/2\sigma^2}$$

Two Gaussian lead to Rayleigh distributions for amplitude

$$A = \sqrt{R^2 + I^2} = \text{amplitude}$$

Super Random – Rayleigh Distribution

- $$P(A) = \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} \quad (0 < A < \infty) \quad \text{(Rayleigh for amplitude)}$$
  $$P(E) = \frac{1}{2\sigma^2} e^{-E/2\sigma^2} \quad (0 < E < \infty) \quad \text{(Rayleigh for Intensity } (E =$$

The K distribution

If Rayleigh is not good enough: introduce new parameters
Many "derivations" – but the primary one is

$$P(E) = \frac{1}{\langle E \rangle} e^{-E/\langle E \rangle} = \frac{1}{z} e^{-E/z} \qquad z = \langle E \rangle$$

Make believe that it is a conditional distribution and write

$$P(E|z) = \frac{1}{z} e^{-E/z}$$

Therefore – a general $P(E)$ – is obtained by averaging over a chosen $P(z)$

$$P(E) = \int P(E|z) P(z) dz$$

One can chose anything one wants for $P(z)$

Scintillation index

$$P(E) = \frac{1}{2\sigma^2} e^{-E/2\sigma^2} = \frac{1}{\langle E \rangle} e^{-E/\langle E \rangle}$$

$$\langle E \rangle = \int EP(E)dE = \int \frac{1}{2\sigma^2} E e^{-E/2\sigma^2} dE = 2\sigma^2$$

$$\langle E^n \rangle = n! \langle E \rangle^n$$

$$\frac{\langle E^n \rangle}{\langle E \rangle^n} = n! \qquad \frac{\langle E^2 \rangle}{\langle E \rangle^2} = 2$$

$$\text{Scintillation Index (for general case)} = \frac{\langle E^2 \rangle}{\langle E \rangle^2} - 1$$

$$\text{For Rayleigh } = 1$$

Is SI time dependent? – it should be.

What it doesn't take into account

$$s = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n e^{iS_n} \qquad \text{random phasor sum}$$

- Assumes no "correlation" between amplitude and phase
- Phase is uniform between 0 and 2π
- Properties of the inducing pulse
- Correlation between the elementary signals
- Everything is "stationary"
- No position or time considerations
- No propagation effects regarding the inducing pulse
- No propagation effects of the induced noise field
- Only one parameter – maybe two

Shot Noise

$$s(t)=\sum_{n=1}^{N}\delta(t-t_n)$$

Is it still shot noise:

- if pulses are broad? – instead of a delta functions
- if there is dispersion during propagation

The History of Noise

On the 100th anniversary of its birth

Leon Cohen

Noise had a glorious birth. It was one of the three miracles of the miracle year, 1905. Einstein, always aiming to solve the greatest of problems and to solve them simply, saw that noise could be the instrument to establish one of the greatest ideas of all

SHOT AND THERMAL NOISE: SCHOTTKY, JOHNSON, AND NYQUIST

The common view that Schottky discovered shot noise and Johnson discovered thermal noise is not correct. Walter Schottky was a physicist who interacted with some of the great scientists of his time, such as Planck. Schottky's range of contributions is extraordinary, both in theory and experiment, and many effects carry his name. We have Schottky diodes, the Shottky barrier, the Shottky effect, among others.

Germany. Passing away at the age of 90, Schottky lived to see what he had wrought.

Schottky wrote his thesis on special relativity but was equally facile in vacuum tubes. He realized that the vacuum tube was the device that would revolutionize the world. He not only made fundamental contributions to vacuum tubes but to many other fields as well. Schottky, in a milestone paper in 1918, was the first to consider what is now called shot noise and thermal noise [17]. We emphasize that he considered *both*. What is now called shot noise is called such not because it is named after him. Schottky called it "shroteffekt," which in German means shot effect, with shot meaning pellets, like gunshot; hence, "shot" noise.

Historically: Two types of waves:

A. Standing waves (periodic waves)

B. pulses
wave groups
transient waves
progressive waves
wave packets
non-recurrent waves
travelling waves
non-periodic waves

Pulse propagation with dispersion

Dispersion relation and group velocity

$$\omega = \omega(k) \qquad v(k) = \frac{\partial \omega(k)}{\partial k}$$

Moments:

$$\langle x^n \rangle_t = \int x^n \, |u(x,t)|^2 \, dx$$

The first moment and standard deviation are

$$\langle x \rangle_t = \langle x \rangle_0 + Vt$$
$$\sigma^2_{x|t} = \sigma^2_{x|0} + 2\,t\,\mathrm{Cov}_{xv|0} + t^2 \sigma^2_v$$

where

$$V = \langle v \rangle_0 = \int v(k) \, |S(k,0)|^2 \, dk = \langle v \rangle_0$$

$$\sigma^2_v = \int (v(k) - V)^2 \, |S(k,0)|^2 \, dk$$

$$\mathrm{Cov}_{xv|0} = \tfrac{1}{2} \langle v\mathcal{X} + \mathcal{X}v \rangle_0 - \langle v \rangle_0 \langle x \rangle_0$$

A simple FOM model with dispersion 1/4

Dispersion relation:

$$\omega(k) = ck + \gamma k^2/2$$

Initial pulse:

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x}$$

Exact time dependent solution ($\eta = \alpha - i\beta$)

$$u(x,t) = \frac{(\alpha/\pi)^{1/4}}{\sqrt{1+i\eta\gamma t}} \exp\left[-\frac{\eta}{2}\frac{(x-ct-k_0\gamma t)^2}{1+i\gamma\eta t} + ik_0(x-ct) - i\gamma k_0^2 t/2\right]$$

In terms of amplitude and phase: next slide

A simple FOM model with dispersion 2/4

Initial pulse:

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x}$$

Exact time dependent solution in terms of amplitude and phase:

$$u(x,t) = \frac{1}{(2\pi\sigma_{x|t}^2)^{1/4}} \exp\left[-\frac{(x-ct-k_0\gamma t)^2}{4\sigma_{x|t}^2}\right] \times$$
$$\exp\left[i\frac{(x-ct-k_0\gamma\, t)^2\,\{\beta+\gamma(\alpha^2+\beta^2)t\}}{4\alpha\sigma_{x|t}^2} + ik_0(x-ct) - i\gamma k_0^2\, t/2 \right. \tag{1}$$

where

$$\sigma_{x|t}^2 = \frac{1}{2\alpha}\left[1+2\beta\gamma t+\gamma^2(\alpha^2+\beta^2)\,t^2\right] \tag{2}$$

$$; \qquad \delta = \frac{1}{2}\arctan\frac{\gamma\alpha t}{1+\gamma\beta t} \tag{3}$$

"Correlation" between phase and amplitude – an example 3/4

Dispersion relation $\omega(k) = ck + \gamma k^2/2$

initial pulse: $u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x}$

Initially:

Phase: $\beta x^2/2 + k_0 x$

Amplitude: $(\alpha/\pi)^{1/4} e^{-\alpha x^2/2}$

- No correlation – α, β, k_0 appear separately in phase and amplitude
- Of course the parameters c, γ of the dispersion relation do not appear.

"Correlation" between phase and amplitude – an example 4/4

Later:

Phase: $$\frac{(x-ct-k_0\gamma\, t)^2\,\{\beta+\gamma(\alpha^2+\beta^2)t\}}{2[1+2\beta\gamma t+\gamma^2(\alpha^2+\beta^2)\,t^2]} + k_0(x-ct) - \gamma k_0^2\, t/2$$

$$-\frac{1}{2}\arctan\frac{\gamma\alpha t}{1+\gamma\beta t}$$

Amplitude: $$\frac{(\alpha/\pi)^{1/4}}{[1+2\beta\gamma t+\gamma^2(\alpha^2+\beta^2)\,t^2]^{1/4}}\exp\left[-\frac{\alpha(x-ct-k_0\gamma t)^2}{2[1+2\beta\gamma t+\gamma^2(\alpha^2+\beta^2)\,t^2]}\right]$$

Highly correlated: $\alpha, \beta, k_0, c, \gamma$ appear in both phase and amplitude

Spread in time:

$$\langle\, x \,\rangle_t = (c + \gamma k_0)\, t$$

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2 + i\beta x^2/2 + i k_0 x}$$

$$\sigma^2_{x|t} = \sigma^2_{x|0}\left[1 + 2\beta\gamma t + \gamma^2\left(\frac{1}{4\sigma^4_{x|0}} + \beta^2\right) t^2\right]$$

Depends on the initial spread in a non-linear way – Always increases if $\beta = 0$

$\sigma^2_{x|t}$ —— Can increase in two fundamentally different ways – one is "paradoxical"

Special case:

$\beta = 0,\ k_0 = 0$

$$u(x,0) = (\alpha/\pi)^{1/4}\, e^{-\alpha x^2/2}$$

$$u(x,t) = \frac{1}{(2\pi\sigma_{x|t}^2)^{1/4}} \exp\left[-\frac{(x-ct)^2}{4\sigma_{x|t}^2}\right] \times \exp\left[i\frac{\gamma\alpha t(x-ct)^2}{4\sigma_{x|t}^2}\right]$$

$$\sigma_{x|t}^2 = \sigma_{x|0}^2\left[1 + \gamma^2 \frac{1}{4\sigma_{x|0}^4}\, t^2\right]$$

Random:The Wigner Spectrum

An ensemble average of the Wigner distribution

$$\overline{W}_X(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[X^*(t-\tau/2)X(t+\tau/2)\right]e^{-i\tau\omega}d\tau$$

where $X(t)$ is a *nonstationary* random process

Noise:The Wigner Spectrum

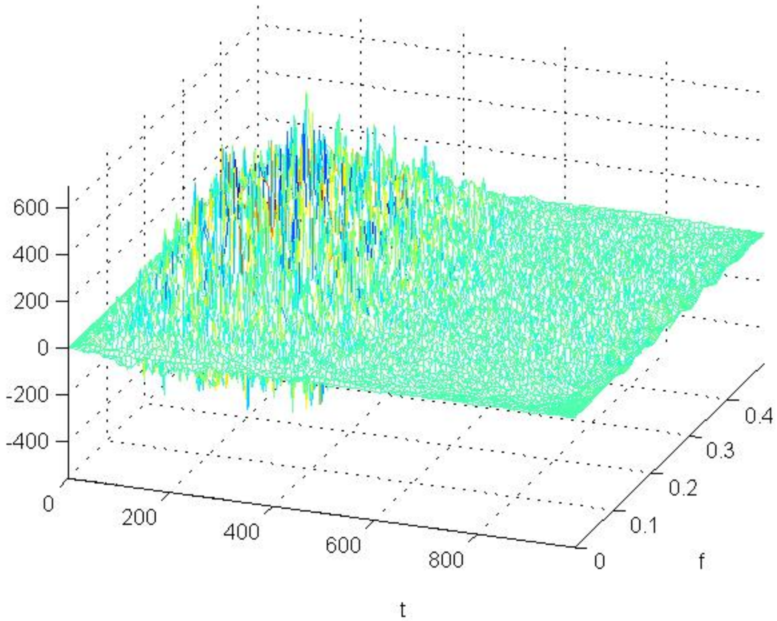
$$\overline{W}(t,\omega) = \frac{1}{2\pi}\int E[u^*(t-\tfrac{1}{2}\tau)u(t+\tfrac{1}{2}\tau)]e^{-i\tau\omega}d\tau$$

auto correlation function:

$$E[u^*(t-\tfrac{1}{2}\tau)u(t+\tfrac{1}{2}\tau)] = \int \overline{W}(t,\omega)e^{i\tau\omega}d\omega$$

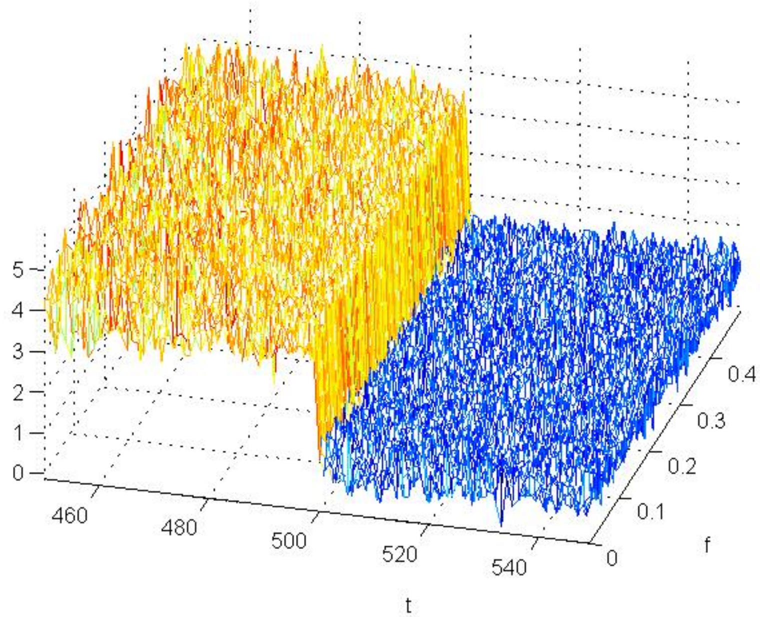
$$E[u^*(t_1)u(t_2)] = \int \overline{W}(\frac{t_1+t_2}{2},\omega)e^{i(t_2-t_1)\omega}d\omega$$

Time-varying White noise: Step variation



Wigner distribution

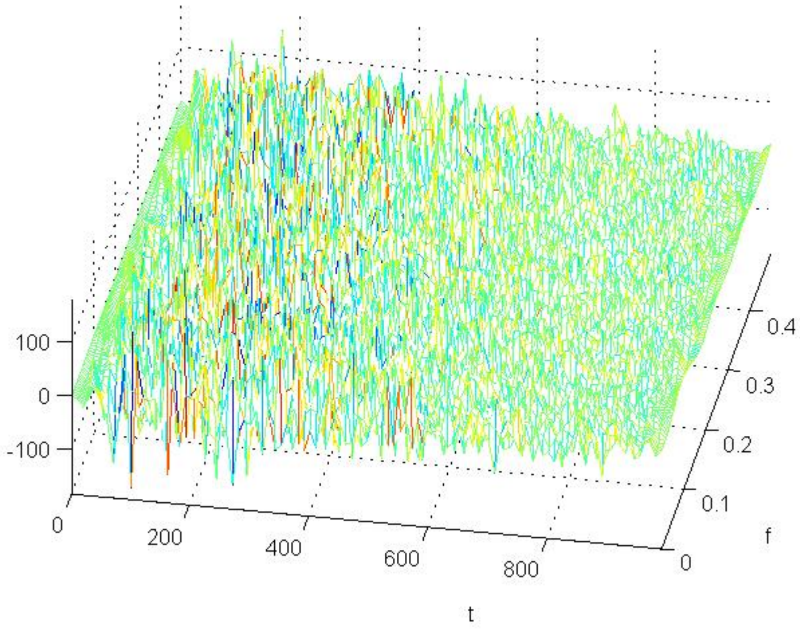
(One realization)



Wigner spectrum

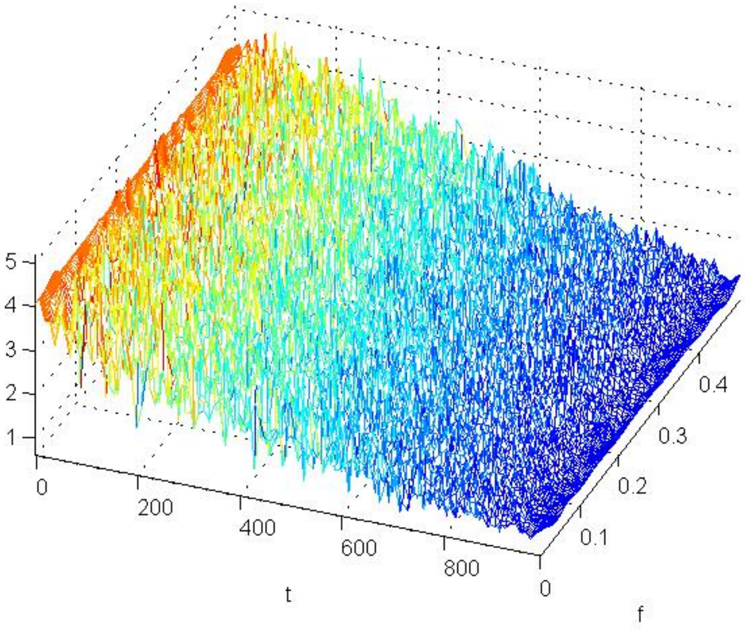
(10^5 realizations)

Time-varying White noise: Linear variation



Wigner distribution

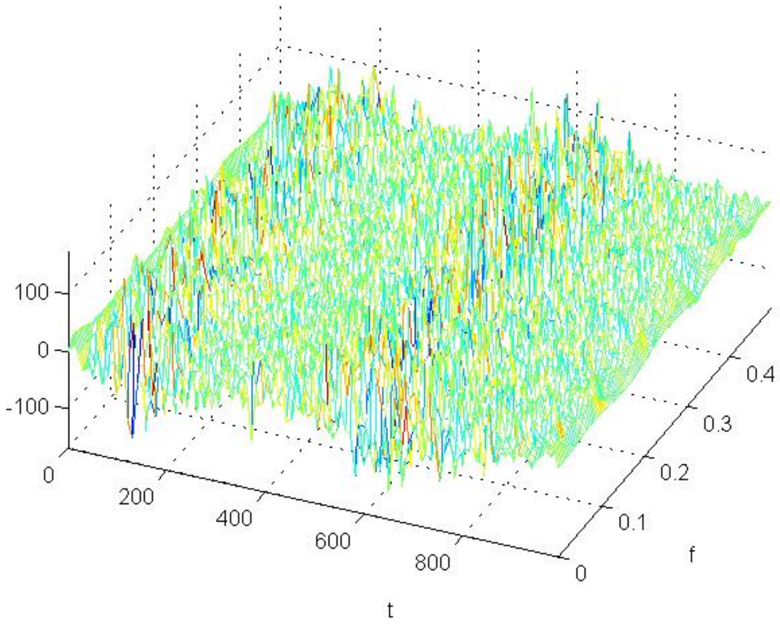
(One realization)



Wigner spectrum

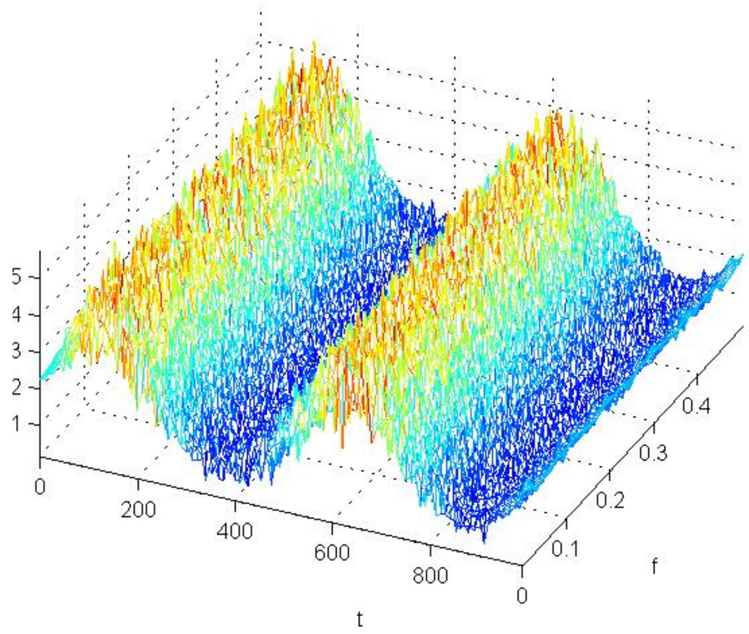
(10^5 realizations)

Time-varying White noise: Cyclic variation



Wigner distribution

(One realization)



Wigner spectrum

(10^5 realizations)

Time dependent autocorolation function

$$u_n(x,t) = (\alpha/\pi)^{1/4} \exp\left[-(\alpha - i\beta)(x - x_n - ct)^2/2 + ik(x - x_n - ct)\right]$$

$$P(x_n) = \sqrt{\frac{q/2}{\pi}} e^{-q(x_n - z)^2/2}$$

$$\langle u_n(x, t+\tau/2) u_n^*(x, t-\tau/2) \rangle = \int u_n(x, t+\tau/2) u_n^*(x, t-\tau/2) P(x_n) dx_n$$

$$= \sqrt{\frac{q\alpha}{\pi(2a+q)}} \exp[-\alpha c^2 \tau^2/4 + i\beta c(ct - x)\tau - ikc\tau]$$

$$\exp\left[-\frac{\beta^2 c^2 \tau^2/4 + \alpha q(x - ct - z)^2/2 - i\beta c\tau(\alpha(x - ct) + qz/2)}{\alpha + q/2}\right]$$

Wigner Spectrum

$$u_n(x,t) = (\alpha/\pi)^{1/4} \exp\left[-(\alpha - i\beta)(x - x_n - ct)^2/2 + ik(x - x_n - ct)\right]$$

$$P(x_n) = \sqrt{\frac{q/2}{\pi}} e^{-q(x_n - z)^2/2}$$

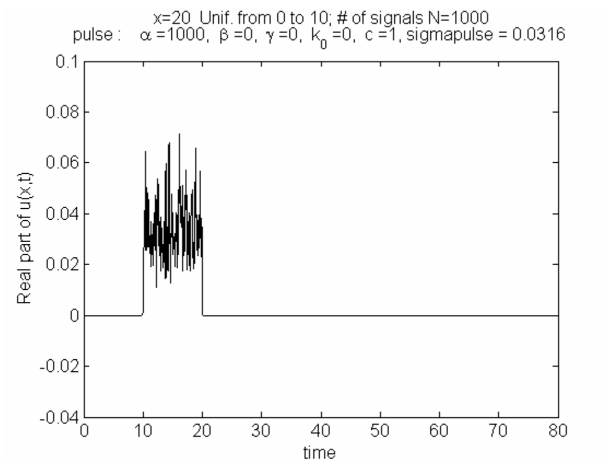
$$d = x - ct - z$$
$$e = \omega/c + k_0$$

$$W(t,\omega) = \frac{1}{\pi c}\sqrt{\frac{\alpha q/2}{\alpha^2 + \beta^2 + \alpha q/2}}$$

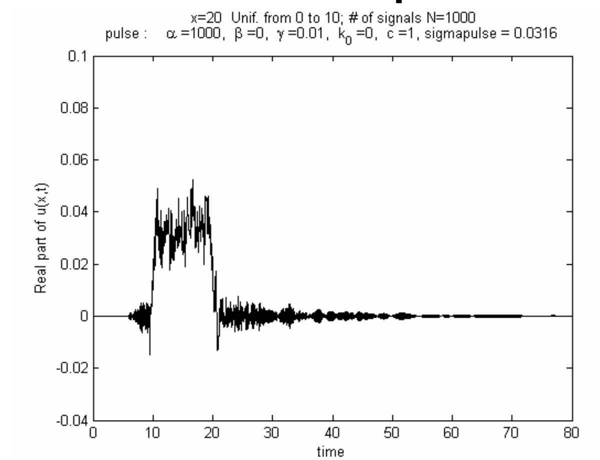
$$\exp\left[-\frac{e^2(\alpha + q/2) + (\alpha^2 + \beta^2)(q/2)d^2 + 2\beta q e d}{\alpha^2 + \beta^2 + \alpha q/2}\right]$$

Noise propagation with dispersion

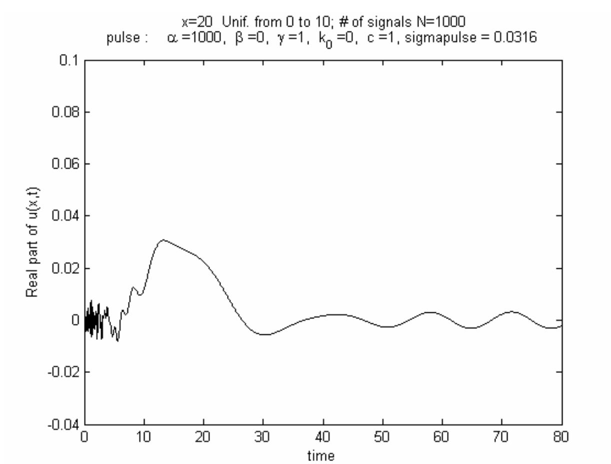
No dispersion:



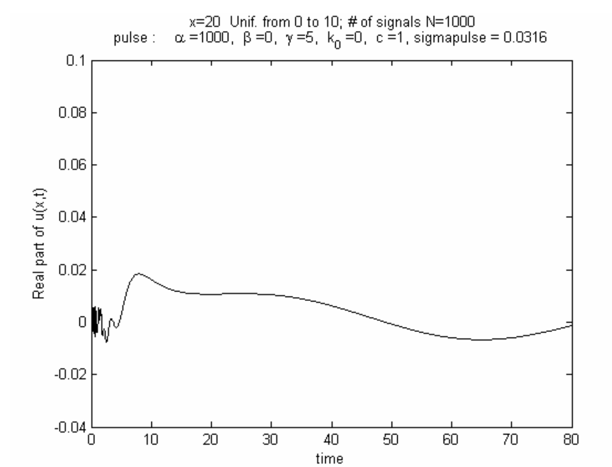
A bit of dispersion:



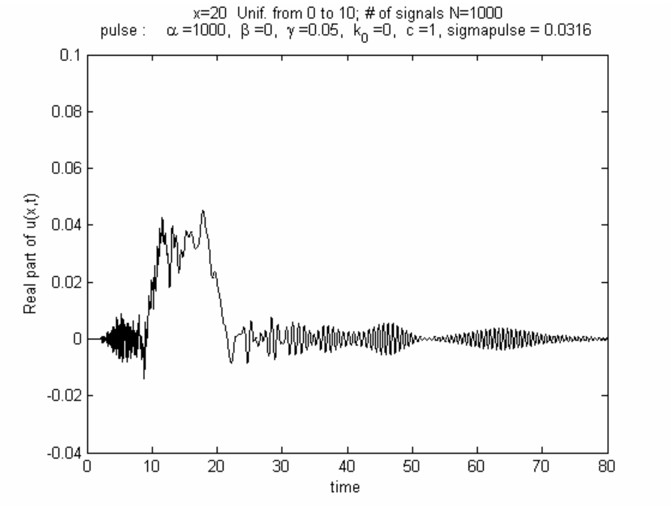
more:



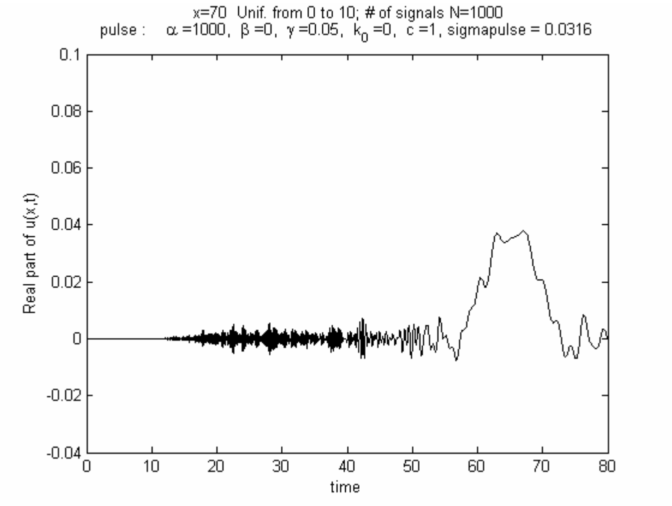
even more:



Noise with dispersion – distance dependence

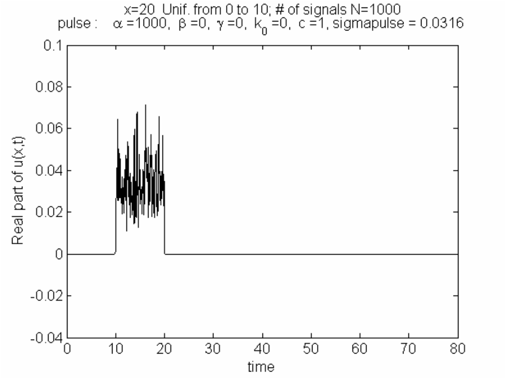


X=20

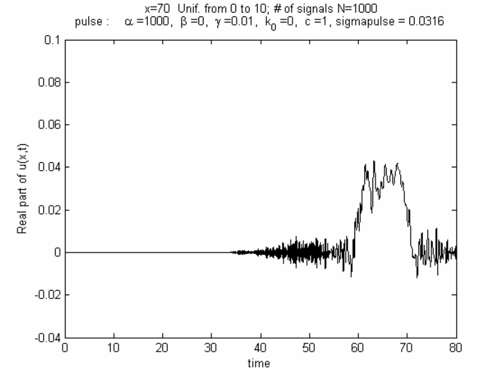


X=70

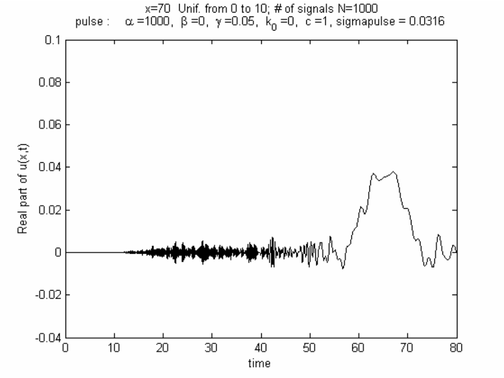
Noise with dispersion – distance dependence



No dispersion

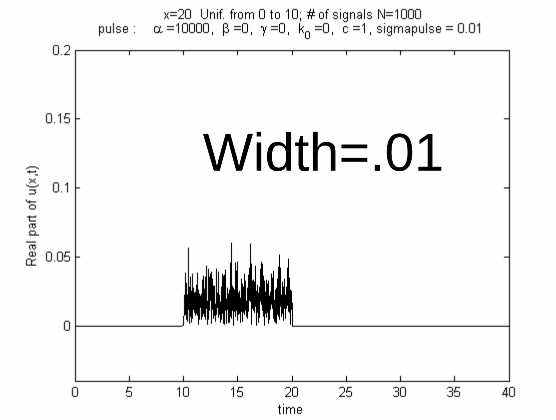


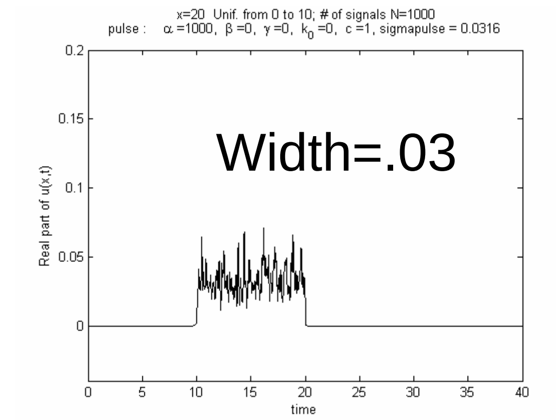
A bit

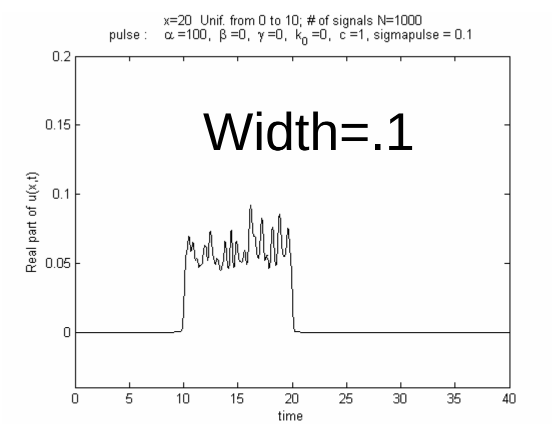


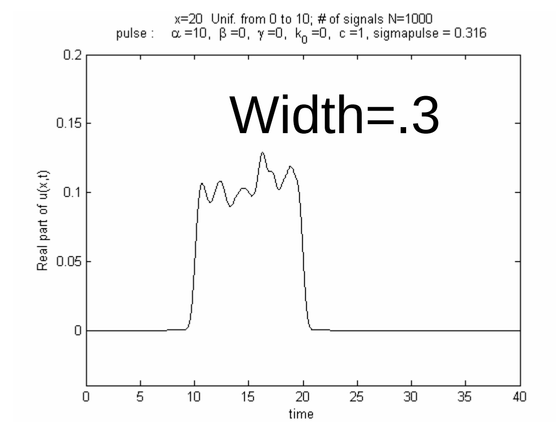
more

No dispersion – varying width of pulses

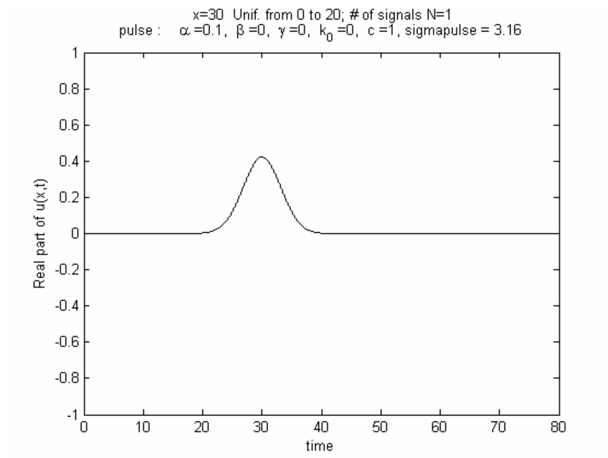


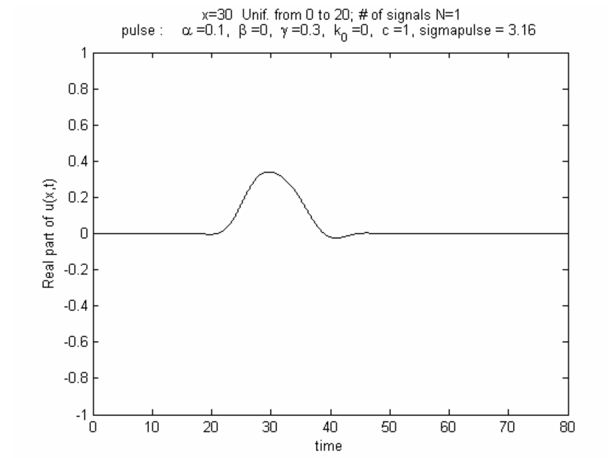


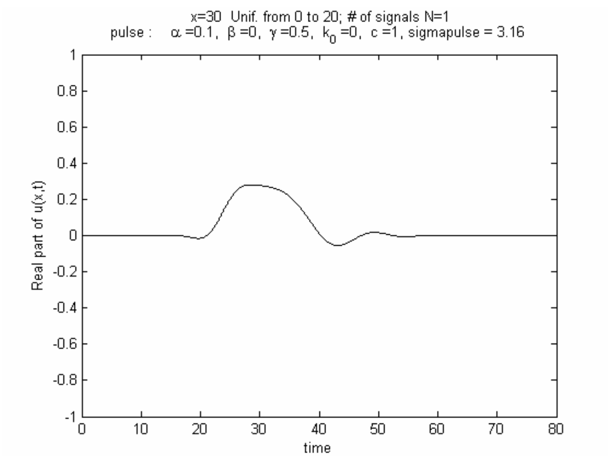


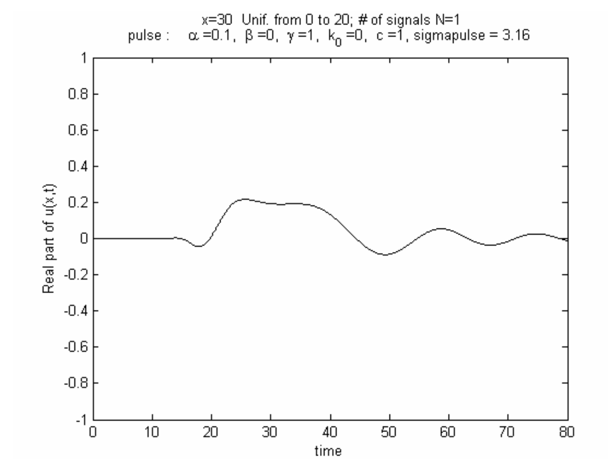


One pulse-with dispersion

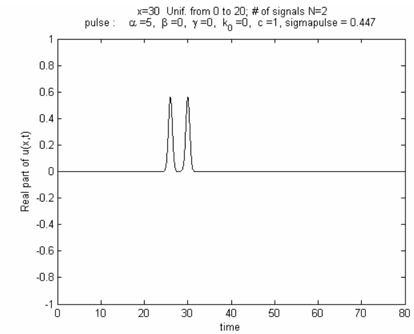


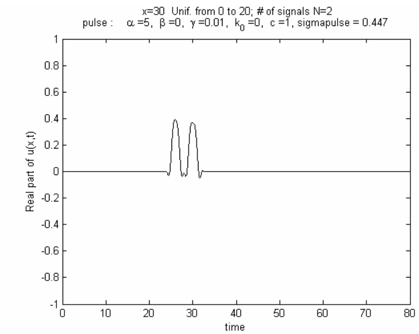


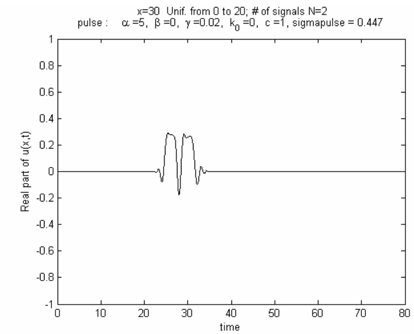


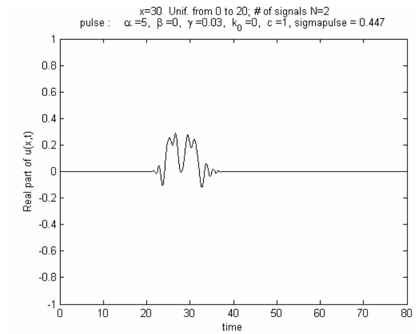


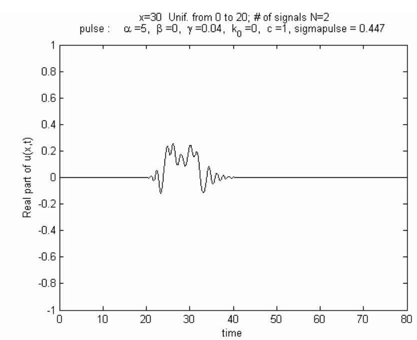
Two pulses-with dispersion

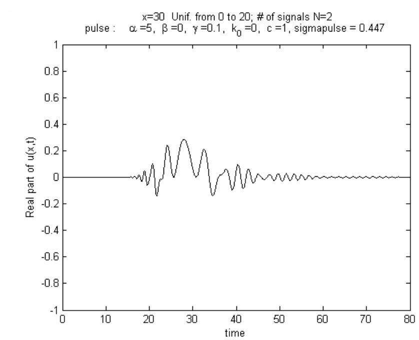


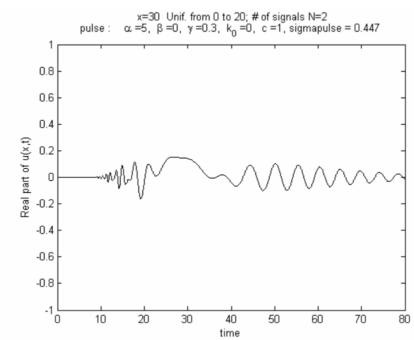


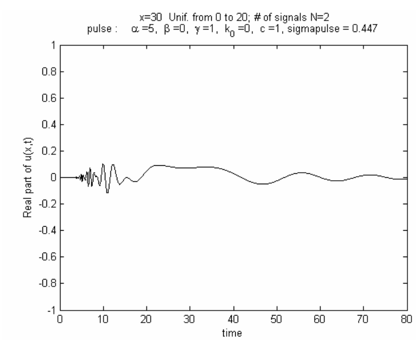






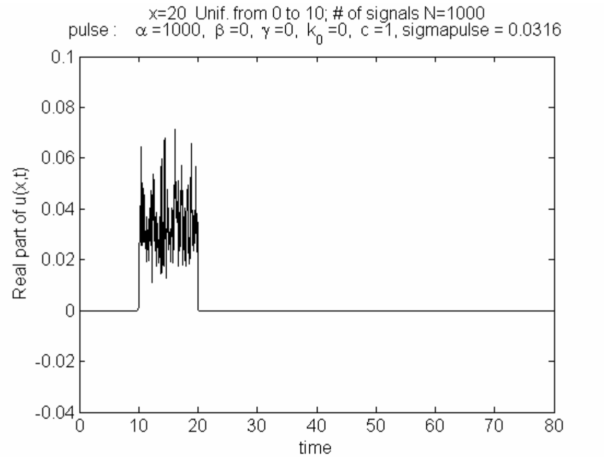




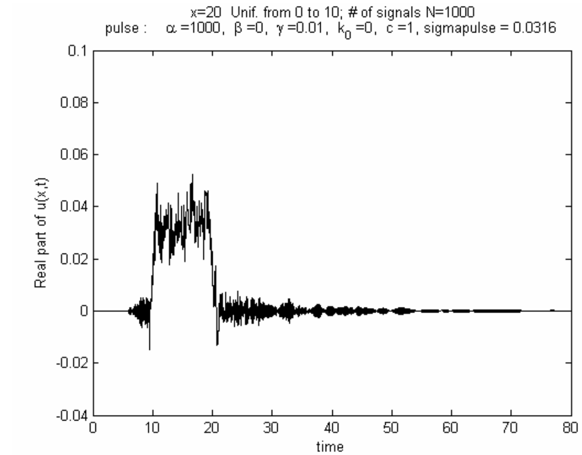


Noise propagation with dispersion

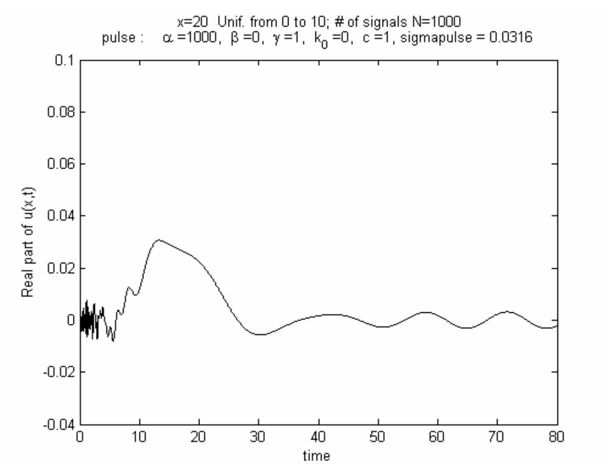
No dispersion:



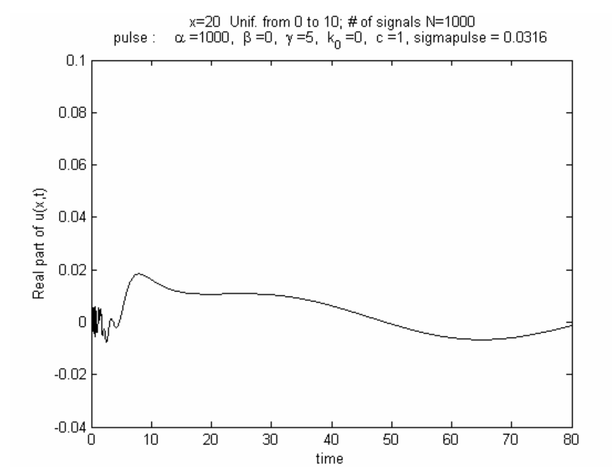
A bit of dispersion:



more:

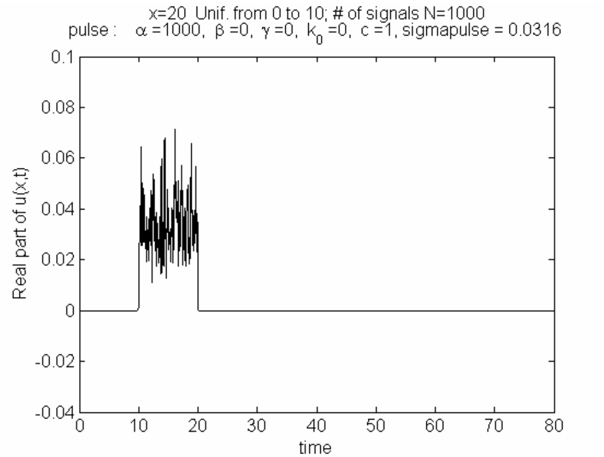


even more:

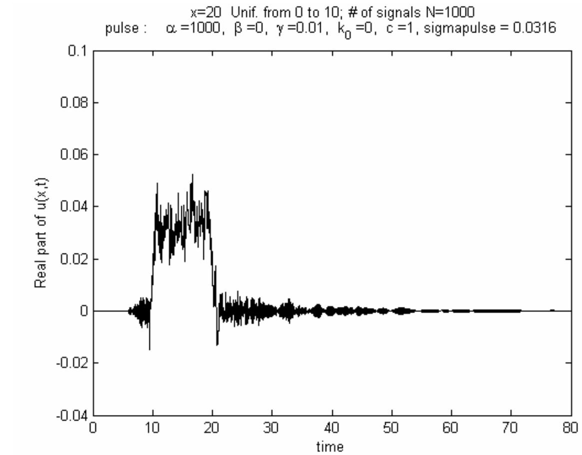


Noise propagation with dispersion

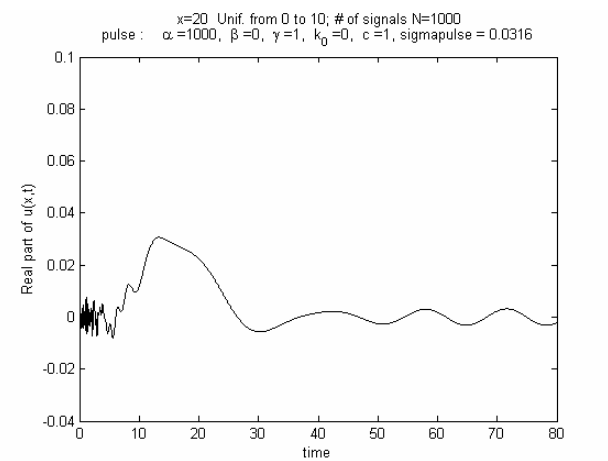
No dispersion:



A bit of dispersion:



more:



even more: